

A New Method for Solving Fuzzy Bernoulli Differential Equation

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Abstract. In this paper, we introduce a new method for solving fuzzy Bernoulli differential equation (FBDE) under generalized differentiability. At the beginning, by stating the theorems, we define n^{th} power of LR fuzzy function and the derivative of LR fuzzy function. Then, we obtain core function to determine LR fuzzy solution, through solving 1-cut FBDE, and calculate spread functions by finding the sign of real valued functions of coefficients of FBDE and finding the sign of core function. Also, numerical examples are presented to verify the effectiveness of the proposed method.

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1 Introduction

Mathematical modeling of many laws in nature that are associated with change, especially in the field of medicine, such as diseases caused by

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covid-19 virus and HIV, etc., lead to the formation of differential equations ([3,6,13,14,18,23]) And whenever the variables in problem are qualitative and ambiguous, we are dealing with fuzzy differential equations ([2,17,22,24,33]). The key tool in fuzzy differential equations is the fuzzy derivative. The concept of fuzzy derivative was first introduced in 1972 by Chang and Zadeh ([26]). In the following, Dubois and Prade derivative ([8]) Puri and Ralescu derivative ([20]) and Seikkala derivative ([27]) and Kandel-Friedman-Ming derivative ([19]) and Hüllermeier derivative ([9]) were introduced. In [5], the study of strongly generalized differentiability of fuzzy-number-valued functions was discussed. Given the comprehensiveness of this derivative and the fact that it does not have the drawbacks of previous derivatives, researchers have used it extensively in solving differential equations.

Numerical methods such as the fuzzy Euler method, predictor–corrector method, Taylor method, Runge-Kutta method, Nyström method in ([1,10,12,21,30,31,32,34]), Adomian semi-analytical method in ([16,28]) and Laplace analytical method in ([29]) for fuzzy differential equations under the Hukuhara differentiability concept was presented.

One of the kinds of fuzzy first order differential equation is fuzzy Bernoulli differential equation which has many functions in financial mathematics. In ([25]), the Picard method to solve the fuzzy first order differential equation as follows:

$$\begin{aligned}\tilde{u}'(t) &= \tilde{Q}(t)\tilde{u}(t) + \tilde{R}(t)\tilde{u}^n(t), \quad 0 \leq t \leq T, \quad T \in \mathfrak{R} \\ \tilde{u}(t) &= \tilde{a}_0,\end{aligned}$$

where $\tilde{Q}(t)$, $\tilde{R}(t)$ and $\tilde{u}(t)$ are fuzzy functions and \tilde{a}_0 is fuzzy constant value, is proposed and this differential equation was mistakenly called the Bernoulli fuzzy differential equation, but with regard to operation on LR fuzzy number, it is not a Bernoulli differential equation and it is a first-order fuzzy differential equation.

Now, given that no research has been done on Bernoulli's differential equations so far, in the present paper, a new simple and effective method is proposed in which the solution of fuzzy Bernoulli differential equation is determined using operations on LR fuzzy numbers and generalized Hukuhara difference and derivative.

The rest of the paper is as follows: Basic necessary definitions are pre-

sented in section 2. The proposed method for solving fuzzy differential equation is introduced in section 3. Numerical examples are involved in section 4 and finally conclusion is presented in section 5.

2 Basic definitions

Definition 2.1. ([11]) *A fuzzy number \tilde{M} is called LR-type if there exist reference functions L (for left), R (for right), and scalars $\alpha > 0$, $\beta > 0$ such that:*

$$\mu_{\tilde{M}}(x) = \begin{cases} L\left(\frac{m-x}{\alpha}\right) & \text{for } x \leq m, \\ R\left(\frac{x-m}{\beta}\right) & \text{for } x \geq m. \end{cases}$$

where m is a real number that is said to the mean value of \tilde{M} . α and β denote the left and the right spread, respectively. Symbolically, \tilde{M} is described by (m, α, β) . The function $L(\cdot)$, which is called left shape function satisfies:

- (1) $L(x) = L(-x)$,
- (2) $L(0) = 1$ and $L(1) = 0$,
- (3) $L(x)$ is non increasing on $[0, \infty)$.

The definition of a right shape function $R(\cdot)$ is similar to that of $L(\cdot)$.

Definition 2.2. ([7,11]) *Consider \tilde{M} , \tilde{N} as two fuzzy numbers of LR-type:*

$$\tilde{M} = (m, \alpha, \beta), \quad \tilde{N} = (n, \gamma, \delta)$$

Then

$$\begin{aligned} (m, \alpha, \beta) + (n, \gamma, \delta) &= (m + n, \alpha + \gamma, \beta + \delta), \\ \lambda (m, \alpha, \beta) &= \begin{cases} (\lambda m, \lambda \alpha, \lambda \beta) & \lambda \geq 0, \\ (\lambda m, -\lambda \beta, -\lambda \alpha) & \lambda < 0, \end{cases} \\ (m, \alpha, \beta) \odot (n, \gamma, \delta) &\approx \begin{cases} (m n, m \gamma + n \alpha, m \delta + n \beta) & \tilde{M}, \tilde{N} > 0, \\ (m n, n \alpha - m \delta, n \beta - m \gamma) & \tilde{M} < 0, \tilde{N} > 0, \\ (m n, m \gamma - n \beta, m \delta - n \alpha) & \tilde{M} > 0, \tilde{N} < 0, \\ (m n, -n \beta - m \delta, -n \alpha - m \gamma) & \tilde{M}, \tilde{N} < 0. \end{cases} \end{aligned}$$

Definition 2.3. ([4]) *The generalized Hukhara difference of two fuzzy numbers is determined as follow:*

$$u -_{gH} v = w \Leftrightarrow \begin{cases} (i) & u = v + w, \\ \text{or} & (ii) v = u + (-1) w. \end{cases}$$

Definition 2.4. ([15]) *If $x_0 \in (a, b)$ and h is such that $x_0 + h \in (a, b)$, then the gH -derivative of a function $f : (a, b) \rightarrow I$ can be described as follow:*

$$f'_{gH}(x_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + h) -_{gH} f(x_0)}{h}.$$

3 Solving fuzzy Bernoulli differential equation

Among various important problems in the field of mathematics, solving differential equations especially fuzzy Bernoulli differential equations can be inferred as one of the most important problems. Therefore, in the following a new method is introduced to solve such problems. The proposed method is shown to be simple and practical in use.

Definition 3.1. *The function $\tilde{u}(t)$ is called LR fuzzy function if its output are LR fuzzy numbers for each $t \in R$. The function is shown by $\tilde{u}(t) = (x(t), y(t), z(t))$ where $x(t)$ is the core function and $y(t)$ and $z(t)$ are the left and the right spread functions, respectively. Moreover, it can be said that the fuzzy function $\tilde{u}(t)$ is positive (negative) if $x(t) > 0$ ($x(t) < 0$).*

Definition 3.2. *Consider the differential equation as*

$$\tilde{u}'(t) + p(t) \tilde{u}(t) = q(t) \tilde{u}^n(t), \quad t \geq 0, t \in R \quad (1)$$

with the initial value as

$$\tilde{u}(0) = \tilde{u}_0$$

The aforementioned differential equation is said to be fuzzy Bernoulli differential equation (FBDE) where $p(t)$ and $q(t)$ are real continues functions, $\tilde{u}(t)$ denotes a LR fuzzy function, \tilde{u}_0 refers to a LR fuzzy number and $n \neq 0, 1$.

To solve the problem of FBDE (1), the n^{th} power of LR fuzzy function is required to be defined. Therefore, a theorem is defined in the following to determine $\tilde{u}^n(t)$.

Theorem 3.3. Assume that $\tilde{u}(t) = (x(t), y(t), z(t))$ is a LR fuzzy function

a) If $\tilde{u}(t) > 0$, then

$$\tilde{u}^n(t) = (x^n(t), n x^{n-1}(t)y(t), n x^{n-1}(t)z(t)) .$$

b) If $\tilde{u}(t) < 0$, then

$$u^n(t) = \begin{cases} (x^n(t), n x^{n-1}(t) y(t), n x^{n-1}(t) z(t)) & n \text{ odd,} \\ (x^n(t), -n x^{n-1}(t) z(t), -n x^{n-1}(t) y(t)) & n \text{ even.} \end{cases}$$

Proof.

a) With the help of definition 2.2

$$\begin{aligned} \tilde{u}^2(t) &= (x(t), y(t), z(t)) \cdot (x(t), y(t), z(t)) \\ &= (x^2(t), 2x^2(t)y(t), 2x^2(t)z(t)) , \end{aligned}$$

$$\begin{aligned} \tilde{u}^3(t) &= \tilde{u}^2(t) \cdot u(t) \\ &= (x^2(t), 2x^2(t)y(t), 2x^2(t)z(t)) \cdot (x(t), y(t), z(t)) \\ &= (x^3(t), 3x^2(t)y(t), 3x^2(t)z(t)) , \end{aligned}$$

Now, by continuing this procedure to the n^{th} step, the equation in the following can be achieved:

$$\tilde{u}^n(t) = (x^n(t), n x^{n-1}(t)y(t), n x^{n-1}(t)z(t)) .$$

b) The procedure of the proof in part(b) is similar to that in part(a) and hence is omitted.

□

Definition 3.4. Consider the LR fuzzy function $\tilde{u}(t)$

a) $\tilde{u}(t)$ is $[i.gH]$ -differentiable, If $\tilde{u}(t)$ is gH -differentiable and $\tilde{u}(t + h) - {}_{gH}\tilde{u}(t)$ is calculated from part (i) of definition 2.3.

- b) $\tilde{u}(t)$ is $[ii.gH]$ -differentiable, If $\tilde{u}(t)$ is gH -differentiable and $\tilde{u}(t+h) - {}_{gH}\tilde{u}(t)$ is calculated from part (ii) of definition 2.3.

Now, a theorem is introduced that enables us to determine the first order derivative of LR fuzzy function using $[i.gH]$ -differentiability and $[ii.gH]$ -differentiability.

Theorem 3.5. If $\tilde{u}(t) = (x(t), y(t), z(t))$ is a LR fuzzy function:

- a) If $\tilde{u}(t)$ is $[i.gH]$ -differentiable then $\tilde{u}'(t) = (x'(t), y'(t), z'(t))$.
b) If $\tilde{u}(t)$ is $[ii.gH]$ -differentiable then $\tilde{u}'(t) = (x'(t), -z'(t), -y'(t))$.

Proof. The proof of part (b) is described in the following:

$$\begin{aligned}\tilde{u}'(t) &= \lim_{h \rightarrow 0} \frac{u(t+h) - {}_{gH}u(t)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(x(t+h) - x(t), z(t) - z(t+h), y(t) - y(t+h))}{h} \\ &= \lim_{h \rightarrow 0} \left(\frac{x(t+h) - x(t)}{h}, -\frac{z(t+h) - z(t)}{h}, -\frac{y(t+h) - y(t)}{h} \right) \\ &= (x'(t), -z'(t), -y'(t)).\end{aligned}$$

□

Now for solving FBDE problem a theorem is proposed in the following that solves the problem using 1-cut system and determining the sign $x(t), p(t), q(t)$.

Theorem 3.6. The solution of fuzzy Bernoulli differential equation with the initial value (1), $\tilde{u}(t) = (x(t), y(t), z(t))$ is as follows

A)

- A1) $[i.gH]$ -differentiability, $x(t) > 0$ or $x(t) < 0$, n odd, $p(t) > 0$, $q(t) > 0$.
A2) $[i.gH]$ -differentiability, $x(t) < 0$, n even, $p(t) > 0$, $q(t) < 0$.
A3) $[ii.gH]$ -differentiability, $x(t) < 0$, n even, $p(t) < 0$, $q(t) > 0$.

A4) [ii.g]-differentiability, $x(t) > 0$ or $x(t) < 0$, n odd, $p(t) < 0$, $q(t) < 0$.

$$\tilde{u}(t) = \left(\left(e^{(n-1) \int p(t) dt} \left(\int (1-n) q(t) e^{(1-n) \int p(t) dt} dt + c_1 \right) \right)^{\frac{1}{1-n}}, \right. \\ \left. c_2 e^{-\int p(t) dt} e^{n \int q(t) x^{n-1}(t) dt}, c_3 e^{-\int p(t) dt} e^{n \int q(t) x^{n-1}(t) dt} \right),$$

B)

B1) [i.gH]-differentiability, $x(t) < 0$, n even, $p(t) > 0$, $q(t) > 0$.

B2) [i.gH]-differentiability, $x(t) > 0$ or $x(t) < 0$, n odd, $p(t) > 0$, $q(t) < 0$.

B3) [ii.gH]-differentiability, $x(t) > 0$ or $x(t) < 0$, n odd, $p(t) < 0$, $q(t) > 0$.

B4) [ii.gH]-differentiability,

$$x(t) < 0, n \text{ even}, p(t) < 0, q(t) < 0.$$

$$\tilde{u}(t) = \left(\left(e^{(n-1) \int p(t) dt} \left(\int (1-n) q(t) e^{(1-n) \int p(t) dt} dt + c_1 \right) \right)^{\frac{1}{1-n}}, \right. \\ \left. e^{-\int p(t) dt} e^{n \int q(t) x^{n-1}(t) dt} \left(\int -n q(t) x^{n-1}(t) \right. \right. \\ \left. \left. \left(c_2 e^{-\int p(t) dt} e^{-n \int q(t) x^{n-1}(t) dt} \right) \right. \right. \\ \left. \left. \int e^{\int p(t) - n \int q(t) x^{n-1}(t) dt} dt + c_3 \right) \right), \\ \left. e^{-\int p(t) dt} e^{n \int q(t) x^{n-1}(t) dt} \left(\int -n q(t) x^{n-1}(t) \right. \right. \\ \left. \left. \left(c_2 e^{-\int p(t) dt} e^{-n \int q(t) x^{n-1}(t) dt} \right) e^{\int p(t) - n \int q(t) x^{n-1}(t) dt} dt + c_4 \right) \right),$$

C)

C1) *[i.gH]-differentiability*, $x(t) > 0$ or $x(t) < 0$, n odd, $p(t) < 0$, $q(t) > 0$.

C2) *[i.gH]-differentiability*, $x(t) < 0$, n even, $p(t) < 0$, $q(t) < 0$.

C3) *[ii.gH]-differentiability*, $x(t) < 0$, n even, $p(t) > 0$, $q(t) > 0$.

C4) *[ii.gH]-differentiability*, $x(t) > 0$ or $x(t) < 0$, n odd, $p(t) > 0$, $q(t) < 0$.

$$\begin{aligned} \tilde{u}(t) = & \left(\left(e^{(n-1) \int p(t) dt} \left(\int (1-n) q(t) \right. \right. \right. \\ & \left. \left. \left. e^{(1-n) \int p(t) dt} dt + c_1 \right) \right)^{\frac{1}{1-n}}, e^{-\int p(t) dt} e^{n \int q(t) x^{n-1}(t) dt} \right. \\ & \left(\int p(t) \left(c_2 e^{\int p(t) dt} e^{n \int q(t) x^{n-1}(t) dt} \right) \right. \\ & \left. e^{\int p(t) - n \int q(t) x^{n-1}(t) dt} dt + c_3 \right), e^{-\int p(t) dt} e^{n \int q(t) x^{n-1}(t) dt} \\ & \left. \left(\int p(t) \left(c_2 e^{\int p(t) dt} e^{n \int q(t) x^{n-1}(t) dt} \right) e^{\int p(t) - n \int q(t) x^{n-1}(t) dt} dt + c_4 \right) \right), \end{aligned}$$

D)

D1) *[i.gH]-differentiability*, $x(t) < 0$, n even, $p(t) < 0$, $q(t) > 0$.

D2) *[i.gH]-differentiability*, $x(t) > 0$ or $x(t) < 0$, n odd, $p(t) < 0$, $q(t) < 0$.

D3) *[ii.gH]-differentiability*, $x(t) > 0$ or $x(t) < 0$, n odd, $p(t) > 0$, $q(t) > 0$.

D4) *[ii.gH]-differentiability*, $x(t) < 0$, n even, $p(t) > 0$, $q(t) < 0$.

$$\begin{aligned} \tilde{u}(t) = & \left(\left(e^{(n-1) \int p(t) dt} \left(\int (1-n) q(t) e^{(1-n) \int p(t) dt} dt + c_1 \right) \right)^{\frac{1}{1-n}}, \right. \\ & e^{-\int p(t) dt} e^{n \int q(t) x^{n-1}(t) dt} \left(\int (p(t) - n q(t) x^{n-1}(t)) \right. \\ & \left. \left(c_2 e^{\int p(t) dt} e^{-n \int q(t) x^{n-1}(t) dt} \right) \right. \\ & \left. e^{\int p(t) - n \int q(t) x^{n-1}(t) dt} dt + c_3 \right), e^{-\int p(t) dt} e^{n \int q(t) x^{n-1}(t) dt} \\ & \left(\int (p(t) - n q(t) x^{n-1}(t)) \right. \\ & \left. \left(c_2 e^{-\int p(t) dt} e^{-n \int q(t) x^{n-1}(t) dt} \right) e^{\int p(t) - n \int q(t) x^{n-1}(t) dt} dt + c_4 \right) \right). \end{aligned}$$

where the constants in (4-7) can be specified using initial values.

Proof.

- A) In A1 to A4, by substituting $\tilde{u}(t) = (x(t), y(t), z(t))$ in (1) and applying definition 2.3, theorems 3.3 and 3.5, the following equations set is gained:

$$\begin{cases} x'(t) + p(t)x(t) = q(t)x^n(t), \\ y'(t) + p(t)y(t) = nq(t)x^{n-1}(t)y(t), \\ z'(t) + p(t)z(t) = nq(t)x^{n-1}(t)z(t), \end{cases}$$

where the first equation is the Bernoulli differential equation and the second and the third equations are linear first order one. Therefore:

$$\begin{aligned} x(t) &= \left(e^{(n-1) \int p(t) dt} \left(\int (1-n)q(t) e^{(1-n) \int p(t) dt} dt + c_1 \right) \right)^{\frac{1}{1-n}}, \\ y(t) &= c_2 e^{-\int p(t) dt} e^{n \int q(t) x^{n-1}(t) dt}, \\ z(t) &= c_3 e^{-\int p(t) dt} e^{n \int q(t) x^{n-1}(t) dt}, \end{aligned} \quad (2)$$

- B) In B1 to B4, by substituting $\tilde{u}(t) = (x(t), y(t), z(t))$ in (1) and with the help of definition 2.2, theorems 3.3 and 3.5, the following differential equations set can be achieved:

$$\begin{cases} x'(t) + p(t)x(t) = q(t)x^n(t), \\ y'(t) + p(t)y(t) = -nq(t)x^{n-1}(t)z(t), \\ z'(t) + p(t)z(t) = -nq(t)x^{n-1}(t)y(t), \end{cases} \quad (3)$$

likewise, $x(t)$ is achieved from (2) and for calculating $y(t)$ and $z(t)$ the following equation is defined:

$$w(t) = y(t) + z(t), \quad (4)$$

Now, the following equation is obtained through adding the second and the third equation of (3) and substituting $w(t)$:

$$w'(t) + (p(t) + nq(t)x^{n-1}(t))w(t) = 0,$$

As a result:

$$w(t) = c_2 e^{-\int p(t) dt} e^{-n \int q(t) x^{n-1}(t) dt},$$

Now, by substituting (4) in the second equation of (3), the following equation can be achieved:

$$y'(t) + (p(t) - n q(t) x^{n-1}(t)) y(t) = -n q(t) x^{n-1}(t) w(t),$$

Therefore:

$$y(t) = e^{-\int p(t) dt} e^{n \int q(t) x^{n-1}(t) dt} \left(\int (-n q(t) x^{n-1}(t) w(t)) e^{\int p(t) dt} e^{-n \int q(t) x^{n-1}(t) dt} dt + c_2 \right),$$

Similarly, it can be achieved that:

$$z(t) = e^{-\int p(t) dt} e^{n \int q(t) x^{n-1}(t) dt} \left(\int (-n q(t) x^{n-1}(t) w(t)) e^{\int p(t) dt} e^{-n \int q(t) x^{n-1}(t) dt} dt + c_3 \right).$$

For the rest of the cases, the proof can be done similarly. \square

According to theorem 3.6, if the initial value of fuzzy Bernoulli differential equation (1) is symmetric, and FBDE has a solution then it can be easily found that the solution will be a fuzzy symmetric as $u(t) = (x(t), \alpha(t), \alpha(t))$. This case is investigated in example 4.3.

Summary of the method: The procedure of the proposed method for solving equation (1) is briefly described in the following:

In the first step, 1-cut of the system is solved and the sign of $x(t)$, $p(t)$ and $q(t)$ are determined. In the second step, the solution of [i.gH]-differentiable and [ii.gH]-differentiable is obtained using theorem 3.6. In the third step, the solution which is closer to the decision maker is considered as the solution of equation (1). It must be mentioned that the fuzzy Bernoulli differential equation will not have fuzzy solution if for each $t \geq 0$, $y(t)$, $z(t)$ are negative or $y(t) \geq 0$ in $[a, b]$, $z(t) \geq 0$ in $[c, d]$, where $[a, b] \cap [c, d] = \phi$.

4 Numerical examples

In order to show the performance and applicability of the proposed method, some numerical examples are presented in this section. It is

also found that the proposed method is simple in use and application.

Example 4.1. Let a fuzzy Bernoulli differential equation be as follow:

$$u'(t) + t \tilde{u}(t) = 2t \tilde{u}^2(t) \quad \tilde{u}(0) = (3, 2, 1), \quad (5)$$

In the first step, the following equation can be obtained by solving 1-cut of the system:

$$x(t) = 3 (6 - 5e^{\frac{t^2}{2}})^{-1},$$

It can be seen that $x(t)$ is continues and positive in $[0, 0.5]$ and $p(t) > 0, q(t) > 0$, as a result in the second step [i.gH]-differentiability of theorem 3.6 in part A1 is used to have:

$$y(t) = 2e^{\frac{t^2}{2}} (6 - 5e^{\frac{t^2}{2}})^{-2},$$

$$z(t) = e^{\frac{t^2}{2}} (6 - 5e^{\frac{t^2}{2}})^{-2},$$

It is found that $y(t) > 0, z(t) > 0$ in $[0, 0.5]$, therefore; the solution of fuzzy Bernoulli differential equation (5) with [i.gH]-differentiability can be achieved as:

$$\tilde{u}(t) = \left(3 (6 - 5e^{\frac{t^2}{2}})^{-1}, 2e^{\frac{t^2}{2}} (6 - 5e^{\frac{t^2}{2}})^{-2}, e^{\frac{t^2}{2}} (6 - 5e^{\frac{t^2}{2}})^{-2} \right),$$

here, the solution is illustrated in Figure 1.

Similar to the previous case; using [ii.gH]-differentiability of theorem 3.6 in part D3, the following spreads can be obtained in $[0, 0.295]$:

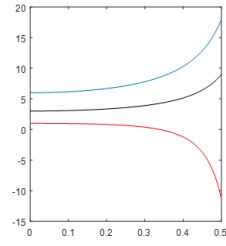
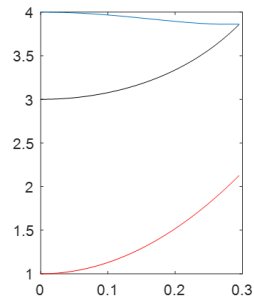
$$y(t) = e^{\frac{t^2}{2}} (6 - 5e^{\frac{t^2}{2}})^{-2} (1944e^{-t^2} - 6480e^{-\frac{t^2}{2}} - 4500e^{\frac{t^2}{2}} + \frac{1875}{2}e^{t^2} + 8100.5),$$

$$z(t) = e^{\frac{t^2}{2}} (6 - 5e^{\frac{t^2}{2}})^{-2} (1944e^{-t^2} - 6480e^{-\frac{t^2}{2}} - 4500e^{\frac{t^2}{2}} + \frac{1875}{2}e^{t^2} + 8099.5),$$

Hence, the solution of fuzzy Bernoulli differential equation (5) using [ii.gH]-differentiability is determined as follows:

$$\tilde{u}(t) = \left(3 (6 - 5e^{\frac{t^2}{2}})^{-1}, e^{\frac{t^2}{2}} (6 - 5e^{\frac{t^2}{2}})^{-2} (1944e^{-t^2} - 6480e^{-\frac{t^2}{2}} - 4500e^{\frac{t^2}{2}} + \frac{1875}{2}e^{t^2} + 8100.5), e^{\frac{t^2}{2}} (6 - 5e^{\frac{t^2}{2}})^{-2} (1944e^{-t^2} - 6480e^{-\frac{t^2}{2}} - 4500e^{\frac{t^2}{2}} + \frac{1875}{2}e^{t^2} + 8099.5) \right).$$

The solution is depicted in Figure 2.

Figure 1: [i.gH]-differentiable solution of equation (5).**Figure 2:** [ii.gH]-differentiable solution of equation (5).

Example 4.2. Assume a fuzzy Bernoulli differential equation as follow:

$$u'(t) - 3t^2 \tilde{u}(t) = -t^2 \tilde{u}^3(t) \quad \tilde{u}(0) = (6, 2, 1), \quad (6)$$

The solution of solving 1-cut of the system is as follow:

$$x(t) = 6e^{t^3} (12e^{2t^3} - 11)^{-\frac{1}{2}},$$

It can be seen that $x(t)$ is positive and continues and $p(t) < 0, q(t) < 0$. Therefore, in the second step, applying [i.gH]-differentiability of theorem 3.6 part D2, the following spreads can be achieved:

$$\begin{aligned} y(t) &= e^{t^3} (12e^{2t^3} - 11)^{-\frac{3}{2}} \left(-\frac{3993}{2}e^{-2t^3} - 7128e^{2t^3} + 2592e^{4t^3} + 6534.5 \right), \\ z(t) &= e^{t^3} (12e^{2t^3} - 11)^{-\frac{3}{2}} \left(-\frac{3993}{2}e^{-2t^3} - 7128e^{2t^3} + 2592e^{4t^3} + 6533.5 \right), \end{aligned}$$

It is found that $y(t) > 0, z(t) > 0$. As a result, the solution of fuzzy Bernoulli differential equation (6) with [i.gH]-differentiability can be calculated as follow:

$$\begin{aligned} \tilde{u}(t) &= \left(6e^{t^3} (12e^{2t^3} - 11)^{-\frac{1}{2}}, e^{t^3} (12e^{2t^3} - 11)^{-\frac{3}{2}} \left(-\frac{3993}{2}e^{-2t^3} - 7128e^{2t^3} \right. \right. \\ &\quad \left. \left. + 2592e^{4t^3} + 6534.5 \right), e^{t^3} (12e^{2t^3} - 11)^{-\frac{3}{2}} \left(-\frac{3993}{2}e^{-2t^3} - 7128e^{2t^3} \right. \right. \\ &\quad \left. \left. + 2592e^{4t^3} + 6533.5 \right) \right), \end{aligned}$$

which is shown in Figure 3. In the same way, the following spreads can be obtained through applying [ii.gH]-differentiability of theorem 3.6 part A4:

$$\begin{aligned} y(t) &= 2e^{t^3} (12e^{2t^3} - 11)^{-\frac{3}{2}}, \\ z(t) &= e^{t^3} (12e^{2t^3} - 11)^{-\frac{3}{2}}, \end{aligned}$$

Therefore, the solution of fuzzy Bernoulli differential equation (6) with [ii.gH]-differentiability can be determined as follow:

$$\tilde{u}(t) = \left(6e^{t^3} (12e^{2t^3} - 11)^{-\frac{1}{2}}, 2e^{t^3} (12e^{2t^3} - 11)^{-\frac{3}{2}}, e^{t^3} (12e^{2t^3} - 11)^{-\frac{3}{2}} \right).$$

The results are presented in Figure 4.

Example 4.3. Let a fuzzy Bernoulli differential equation be as follow:

$$u'(t) + (-t - 1) \tilde{u}(t) = (2t + 2) \tilde{u}^2(t) \quad \tilde{u}(0) = (-4, 3, 3), \quad (7)$$

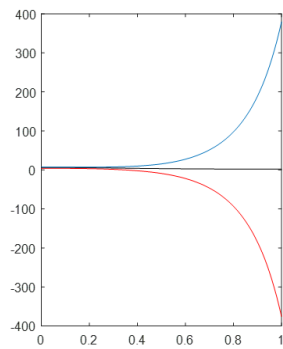
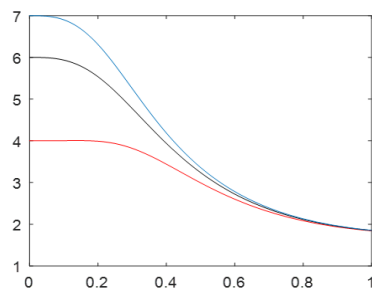
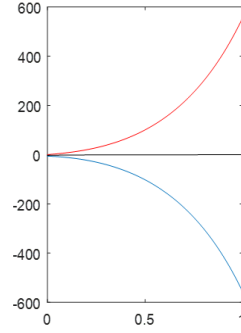
Figure 3: [i.gH]-differentiable solution of equation (6).**Figure 4:** [ii.gH]-differentiable solution of equation (6).

Figure 5: [i.gH]-differentiable solution of equation (7).

Solving 1-cut of the system in the first step results to the following equation:

$$x(t) = 4e^{\frac{t^2}{2}+t} (-8e^{\frac{t^2}{2}+t} + 7)^{-1},$$

It is obvious that $x(t)$ is negative and continues and $p(t) < 0, q(t) > 0$. Hence, in the second step, [i.gH]-differentiability of theorem 3.6 part D1 is used to have the following results:

$$\begin{aligned} y(t) &= e^{\frac{t^2}{2}+t} (-8e^{\frac{t^2}{2}+t} + 7)^{-2} (3e^{-t^2-2t} (-10976e^{\frac{t^2}{2}+t} - 14336e^{\frac{3}{2}t^2+3t} \\ &\quad + 4096e^{2t^2+4t} + 2401) + 56448), \\ z(t) &= e^{\frac{t^2}{2}+t} (-8e^{\frac{t^2}{2}+t} + 7)^{-2} (3e^{-t^2-2t} (-10976e^{\frac{t^2}{2}+t} - 14336e^{\frac{3}{2}t^2+3t} \\ &\quad + 4096e^{2t^2+4t} + 2401) + 56448), \end{aligned}$$

It is found that $y(t) > 0, z(t) > 0$. As a result, the solution of fuzzy Bernoulli differential equation (7) using [i.gH]-differentiability is achieved as follow and is drawn in Figure 5:

$$\begin{aligned} \tilde{u}(t) &= \left(4e^{\frac{t^2}{2}+t} (-8e^{\frac{t^2}{2}+t} + 7)^{-1}, e^{\frac{t^2}{2}+t} (-8e^{\frac{t^2}{2}+t} + 7)^{-2} (3e^{-t^2-2t} \right. \\ &\quad \left. (-10976e^{\frac{t^2}{2}+t} - 14336e^{\frac{3}{2}t^2+3t} + 4096e^{2t^2+4t} + 2401) + 56448), e^{\frac{t^2}{2}+t} \right. \\ &\quad \left. (-8e^{\frac{t^2}{2}+t} + 7)^{-2} (3e^{-t^2-2t} (-10976e^{\frac{t^2}{2}+t} - 14336e^{\frac{3}{2}t^2+3t} + 4096e^{2t^2+4t} \right. \\ &\quad \left. + 2401) + 56448) \right), \end{aligned}$$

As was expected, since the initial condition of equation (7) is sym-

metric, FBDE has fuzzy symmetric solution. Similarly, using [ii.gH]-differentiability of theorem 3.6 part A3, the following results can be obtained:

$$y(t) = 3 e^{\frac{t^2}{2}+t} (- 8e^{\frac{t^2}{2}+t} + 7)^{-1} ,$$

$$z(t) = 3 e^{\frac{t^2}{2}+t} (- 8e^{\frac{t^2}{2}+t} + 7)^{-1} .$$

Moreover, it can be found that fuzzy Bernoulli differential equation (7) does not have [ii.gH]-differentiable solution.

5 Conclusion

Fuzzy Bernoulli differential equations can be considered as one of the applied problems in the field of economic functions and problems related to financial mathematics which due to the fact that these fuzzy equations have been less discussed and no research has been done on them in the present paper, an efficient method is applied to solve fuzzy Bernoulli differential equation in case where the fuzzy variables are LR fuzzy function. It is assumed that the generalized Hukuhara difference and the derivative of the fuzzy function exist. The significant advantage of the proposed method is its simplicity and applicability which enables it to be extended to all fuzzy functions. In the future work, fully fuzzy Bernoulli differential equation are going to be solved and the problem of switching point are going to be considered.

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